Technical Comments

Comment on "Nonzero Free-Free Frequencies of Structures Idealized by Matrix Methods"

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IN a recent Note Newman and Ojalvo[1] discuss the case of free vibration of an unrestrained system. Such systems are also referred to as semidefinite and they are known to possess at least one zero natural frequency, the number of zero frequencies depending on the system.[2] In contrast with the references quoted by the authors, the Note allegedly "develops an alternate method for treating this same problem through direct reduction of the free-free system's stiffness and mass matrices in such a way as to eliminate its zero frequencies." The Note, however, fails to present any information which is not available in the open literature. In fact, virtually the entire material presented in the Note can be found in Ref. 2, and this includes the illustrative example used in the Note. Although it can be argued that Ref. 1 contains in addition a nonsymmetrical formulation, this represents no advantage. On the contrary, a symmetrical formulation is much more convenient for computational purposes, and in this particular case there is no valid reason for attempting to solve a nonsymmetrical eigenvalue problem, because the problem can be symmetrized with ease. In recognition of this fact, Ref. 2 has chosen an approach which bypasses entirely the nonsymmetrical formulation.

References

[1] Newman, M. and Ojalvo, I. V., "Nonzero Free-Free Frequencies of Structures Idealized by Matrix Methods," *AIAA Journal*, Vol. 7, No. 12, Dec. 1969, pp. 2343–2344.

[2] Meirovitch, L., *Analytical Methods in Vibrations*, Macmillan, New York, 1967, pp. 112–115, 263–270.

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Comment on "Nonzero Free-Free Frequencies of Structures Idealized by Matrix Methods"

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IN Ref. 1, Newman and Ojalvo described a method for determining frequencies of unconstrained (free-free) structures through direct reduction of stiffness and mass matrices for the free-free system in such a way as to eliminate its rigid body frequencies. Using the solution of Ref. 1, with boldface symbols to represent matrices, equations of motion for the freely vibrating undamped system can be written as

$$\mathbf{K}_R\mathbf{x}_1 = \omega^2\mathbf{M}_R\mathbf{x}_1;\ \omega^2 \neq 0 \tag{1}$$

where the reduced stiffness and mass matrices are given by

$$\mathbf{K}_R = \mathbf{k}_{11} + \mathbf{k}_{12}\mathbf{R} \tag{2}$$

$$\mathbf{M}_R = \mathbf{m}_{11} + \mathbf{m}_{12}\mathbf{R} \tag{3}$$

and

$$\mathbf{R} = -(\mathbf{T}_{12}{}^T\mathbf{m}_{12} + \mathbf{m}_{22})^{-1}(\mathbf{T}_{12}{}^T\mathbf{m}_{11} + \mathbf{m}_{21}) \tag{4}$$

The subscript 2 refers to the degrees of freedom assigned to eliminate the rigid body motion from the system, whereas the subscript 1 refers to all the remaining degrees of freedom. The transformation matrix $\mathbf{T}_{12}$ is used to express the degrees of freedom $\mathbf{x}_1$ in terms of $\mathbf{x}_2$ for rigid body motion, i.e.,

$$\mathbf{x}_1 = \mathbf{T}_{12}\mathbf{x}_2 \tag{5}$$

The matrix $\mathbf{T}_{12}$ can be generated either from kinematical considerations or from the equations of equilibrium. For details, Ref. 2 may be consulted where the following notation was used: $\mathbf{T}_{12} = \mathbf{T}$ and $\mathbf{R} = \mathbf{T}_0$. It should also be noted that Eq. (1) is of the same form as the stiffness equations for the free-free system

$$\begin{bmatrix}\mathbf{k}_{11}\mathbf{k}_{12}\\ \mathbf{k}_{21}\mathbf{k}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix} = \omega^2\begin{bmatrix}\mathbf{m}_{11}\mathbf{m}_{12}\\ \mathbf{m}_{21}\mathbf{m}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix};\ \omega^2 \geqslant 0 \tag{6}$$

The reasons for deriving Eq. (1) are not necessarily the desirability to have a positive definite stiffness matrix describing the system, as it may be implied from Ref. 1. More often, the reason is that the flexibility matrix $\mathbf{k}_{11}{}^{-1}$ is already available for the frequency calculations, as is the case of either the matrix force method of analysis or experimentally obtained matrix of influence coefficients. Furthermore, it should also be pointed out that very efficient eigenvalue computer programs are now available which can handle singular stiffness matrices and will generate zero frequencies and the associated rigid body modes from Eq. (6).

If the use of flexibility matrices is required then Eq. (1) must be modified. Substituting for $\mathbf{k}_{12}$ from[1]

$$\mathbf{T}_{12} = -\mathbf{k}_{11}{}^{-1}\mathbf{k}_{12} \tag{7}$$

in Eq. (1) we obtain

$$\mathbf{k}_{11}(\mathbf{I} - \mathbf{T}_{12}\mathbf{R})\mathbf{x}_1 = \omega^2(\mathbf{m}_{11} + \mathbf{m}_{12}\mathbf{R})\mathbf{x}_1 \tag{8}$$

Subsequent premultiplication by $(\mathbf{I} - \mathbf{T}_{12}\mathbf{R})^{-1}\mathbf{k}_{11}{}^{-1}/\omega^2$ leads to

$$(1/\omega^2)\mathbf{x}_1 = (\mathbf{I} - \mathbf{T}_{12}\mathbf{R})^{-1}\mathbf{k}_{11}{}^{-1}(\mathbf{m}_{11} + \mathbf{m}_{12}\mathbf{R})\mathbf{x}_1 \tag{9}$$

Equation (9) is of the standard form presented in Ref. 2. It can be further simplified if the mass matrix is diagonal ($\mathbf{m}_{12} = \mathbf{0}$ and $\mathbf{m}_{21} = \mathbf{0}$) so that

$$(1/\omega^2)\mathbf{x}_1 = (\mathbf{I} + \mathbf{T}_{12}\mathbf{m}_{22}{}^{-1}\mathbf{T}_{12}{}^T\mathbf{m}_{11})^{-1}\mathbf{k}_{11}{}^{-1}\mathbf{m}_{11}\mathbf{x}_1 \tag{10}$$

In the past, several authors[3–6] considered the special case for which $\mathbf{m}_{22} = \mathbf{0}$ and $\mathbf{m}_{12} = \mathbf{0}$. Naturally, for this case Eq. (9) can no longer be used because the matrix $\mathbf{R}$ would require inversion of a singular (zero) matrix and therefore a different procedure must be adapted. Starting with the basic equation of motion for the system we have[2]

$$\mathbf{x}_1 - \mathbf{T}_{12}\mathbf{x}_2 = \omega^2\mathbf{k}_{11}{}^{-1}\mathbf{m}_{11}\mathbf{x}_1 \tag{11}$$

Premultiplying Eq. (11) by $\mathbf{T}_{12}{}^T\mathbf{m}_{11}$ and noting that the equilibrium condition for the inertia forces is $\mathbf{T}_{12}{}^T\mathbf{m}_{11}\mathbf{x}_1 = \mathbf{0}$

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we have that

$$\mathbf{x}_2 = -\omega^2(\mathbf{T}_{12}{}^T\mathbf{m}_{11}\mathbf{T}_{12})^{-1}\mathbf{T}_{12}{}^T\mathbf{m}_{11}\mathbf{k}_{11}{}^{-1}\mathbf{m}_{11}\mathbf{x}_1 \qquad (12)$$

Finally, substituting Eq. (12) into (11) and dividing by ω^2 we obtain the standard equation

$$(1/\omega^2)\mathbf{x}_1 = (\mathbf{I} - \mathbf{T}_{12}(\mathbf{T}_{12}{}^T\mathbf{m}_{11}\mathbf{T}_{12})^{-1}\mathbf{T}_{12}{}^T\mathbf{m}_{11})\mathbf{k}_{11}{}^{-1}\mathbf{m}_{11}\mathbf{x}_1 \qquad (13)$$

Assuming that $\mathbf{k}_{11}{}^{-1}$ is available, the only matrix inversion required in Eq. (13) is for the product $(\mathbf{T}_{12}{}^T\mathbf{m}_{11}\mathbf{T}_{12})$ which, for three-dimensional structures, is of the order (6×6). This compares with two inversions in Eq. (9), one for $(\mathbf{T}_{12}{}^T\mathbf{m}_{12} + \mathbf{m}_{22})$ of order (6×6) in $\mathbf{R}$ and another for $(\mathbf{I} - \mathbf{T}_{12}\mathbf{R})$ of order $(n \times n)$, where n is the number of degrees of freedom in $\mathbf{x}_1$. It follows therefore that, in general, if the introduction of a few massless node points in the structure is an acceptable idealization so that $\mathbf{m}_{22} = \mathbf{0}$ and $\mathbf{m}_{12} = \mathbf{0}$ (zero inertia in the direction of $\mathbf{x}_2$) then it would be preferable to compute the nonzero frequencies of a free-free system from Eq. (13).

References

[1] Newman, M. and Ojalvo, I. U., "Nonzero Free-Free Frequencies of Structures Idealized by Matrix Methods," *AIAA Journal*, Vol. 7, No. 12, Dec. 1969, pp. 2343–2344.

[2] Przemieniecki, J. S., *Theory of Matrix Structural Analysis*, 1st ed., McGraw-Hill, New York, 1968, pp. 137–147, 310–320.

[3] Rodden, W. P., "On Vibration and Flutter Analysis with Free-Free Boundary Conditions," *Journal of the Aerospace Sciences*, Vol. 28, No. 1, Jan. 1961, pp. 65–66.

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[6] Berman, J. H. and Sklerov, J., Calculation of Natural Modes of Vibration for Free-Free Structures in Three-Dimensional Space," *AIAA Journal*, Vol. 3, No. 1, Jan. 1965, pp. 158–160.

Reply by Authors to L. Meirovitch and J. S. Przemieniecki

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THE approach presented in Meirovitch's excellent book[1] was originally developed by Mack[2] as cited in our Note. In essence, this method reduces the semidefinite free-free system to a positive definite eigenvalue problem by measuring elastic deformations of the structure from a rigid-body reference position. The singular stiffness matrix is reduced to a positive definite form by eliminating rows and columns corresponding to the rigid-body degrees of freedom and a reduced mass matrix may be obtained by either requiring that the system moments be zero or, equivalently, by imposing orthogonality of the flexible modes with the rigid-body mode shapes. However, as was stated in our note,[3] the eigenvectors of the reduced problem obtained in this manner constitute relative coordinates with respect to the assumed reference position. If the modes are required in inertial coordinates, which is usually the case in aeroelastic dynamic stability studies, an additional coordinate transformation must be performed.

Our Note circumvents this extra computation by employing a simple rigid-body transformation matrix to reduce the free-free system's stiffness and mass matrices in such a way as to obviate the need for employing relative coordinates. Thus, the flexible vibration modes are obtained directly in inertial coordinates as the eigenvectors of the reduced system.

The advantages of a symmetrical formulation, as noted in Ref. 1, are not disputed and, for this reason, were presented as Eq. (17) of our Note. We leave it to the reader to decide for himself whether "virtually the entire material presented in the Note" can be found in Ref. 1.

Turning to Przemieniecki's comments on the relative value of obtaining a nonsingular stiffness matrix, the authors agree that eigenvalue algorithms are available to accommodate singular matrices. However, their use often involves penalties with regard to accuracy or computation time. It should also be noted that many highly reliable and efficient numerical methods require the use of positive definite matrices. This fact, coupled with the increasing popularity of the stiffness method[4] (which does not deliver $k_{11}{}^{-1}$ automatically) over the force method, seems to make our formulation a useful alternative to the existing ones.

Another issue introduced by Przemieniecki, following his Eq. (10), concerns the possibility of a singular mass matrix. As implied in our note, we did not consider this situation as it was assumed that m_{11} and m_{22} were positive definite. However, since this possibility exists and the point has been raised, we would suggest elimination of the noninertial degrees of freedom prior to elimination of the rigid-body modes. A standard procedure for accomplishing this is described in Ref. 5.

In connection with this latter point, it should be noted that the discussor's Eq. (13) still contains all the rigid-body modes (the matrix in braces on the right-hand side of the equation is singular). Thus, as noted earlier, Eq. (13) precludes the use of eigenvalue computational methods which require positive definite matrices.

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References

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[4] MacNeal, R. H., *The NASTRAN Theoretical Manual*, SP-221, Oct. 1969, NASA.

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Comment on "The Eigenvalue Problem for Structural Systems with Statistical Properties"

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Nomenclature

K = stiffness matrix
M = mass matrix
λ_i = ith eigenvalue

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